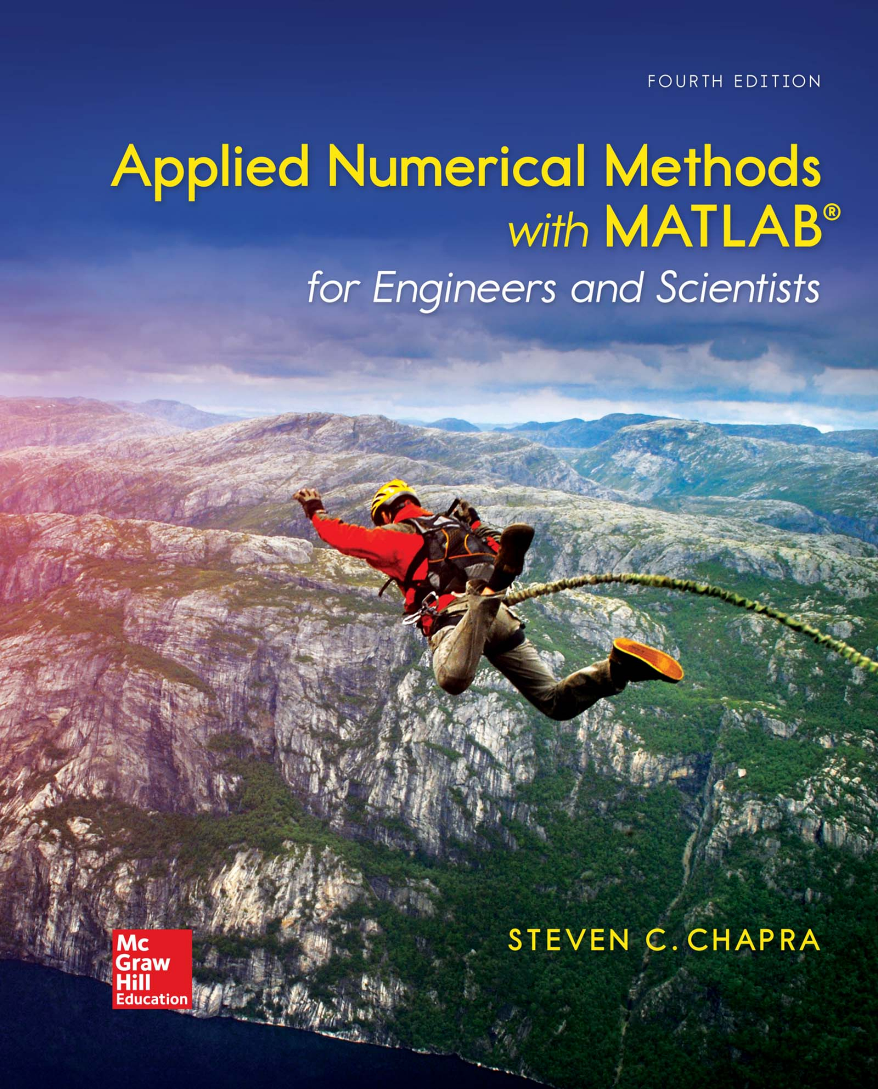
FOURTH EDITION

Applied Numerical Methods with MATLAB®

for Engineers and Scientists



STEVEN C. CHAPRA

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Applied Numerical Methods

with MATLAB® for Engineers and Scientists

Fourth Edition

Steven C. Chapra

Berger Chair in Computing and Engineering
Tufts University

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APPLIED NUMERICAL METHODS WITH MATLAB® FOR ENGINEERS AND SCIENTISTS,
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To

My brothers,
John and Bob Chapra

and

Fred Berger (1947–2015)
who I miss as a good friend, a good man.
and a comrade in bringing the light of engineering
to some of world's darker corners.

ABOUT THE AUTHOR

Steve Chapra teaches in the Civil and Environmental Engineering Department at Tufts University, where he holds the Louis Berger Chair in Computing and Engineering. His other books include *Numerical Methods for Engineers* and *Surface Water-Quality Modeling*.

Steve received engineering degrees from Manhattan College and the University of Michigan. Before joining the faculty at Tufts, he worked for the Environmental Protection Agency and the National Oceanic and Atmospheric Administration, and taught at Texas A&M University and the University of Colorado. His general research interests focus on surface water-quality modeling and advanced computer applications in environmental engineering.

He has received a number of awards for his scholarly contributions, including the Rudolph Hering Medal, the Meriam/Wiley Distinguished Author Award, and the Chandler-Misener Award. He has also been recognized as the outstanding teacher at Texas A&M University (1986 Tenneco Award), the University of Colorado (1992 Hutchinson Award), and Tufts University (2011 Professor of the Year Award).

Steve was originally drawn to environmental engineering and science because of his love of the outdoors. He is an avid fly fisherman and hiker. An unapologetic nerd, his love affair with computing began when he was first introduced to Fortran programming as an undergraduate in 1966. Today, he feels truly blessed to be able to meld his love of mathematics, science, and computing with his passion for the natural environment. In addition, he gets the bonus of sharing it with others through his teaching and writing!

Beyond his professional interests, he enjoys art, music (especially classical music, jazz, and bluegrass), and reading history. Despite unfounded rumors to the contrary, he never has, and never will, voluntarily bungee jump or sky dive.

If you would like to contact Steve, or learn more about him, visit his home page at http://engineering.tufts.edu/cee/people/chapra/ or e-mail him at steven.chapra@tufts.edu.

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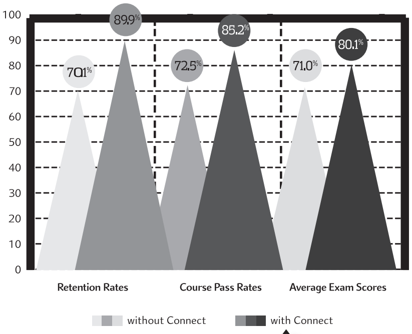

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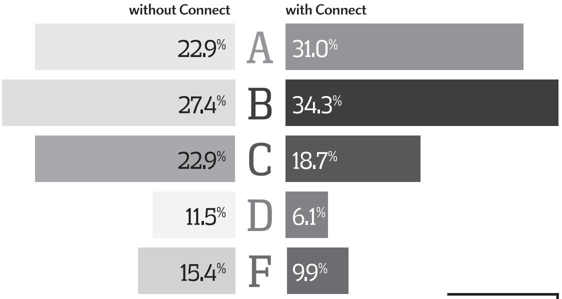

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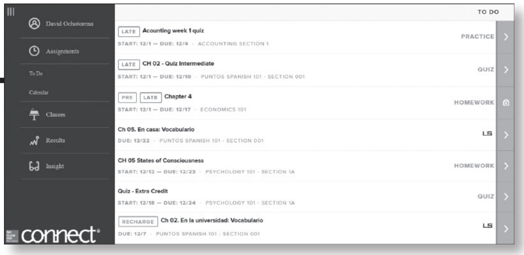

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PREFACE

This book is designed to support a one-semester course in numerical methods. It has been written for students who want to learn and apply numerical methods in order to solve problems in engineering and science. As such, the methods are motivated by problems rather than by mathematics. That said, sufficient theory is provided so that students come away with insight into the techniques and their shortcomings.

MATLAB® provides a great environment for such a course. Although other environments (e.g., Excel/VBA, Mathcad) or languages (e.g., Fortran 90, C++) could have been chosen, MATLAB presently offers a nice combination of handy programming features with powerful built-in numerical capabilities. On the one hand, its M-file programming environment allows students to implement moderately complicated algorithms in a structured and coherent fashion. On the other hand, its built-in, numerical capabilities empower students to solve more difficult problems without trying to "reinvent the wheel."

The basic content, organization, and pedagogy of the third edition are essentially preserved in the fourth edition. In particular, the conversational writing style is intentionally maintained in order to make the book easier to read. This book tries to speak directly to the reader and is designed in part to be a tool for self-teaching.

That said, this edition differs from the past edition in three major ways: (1) new material, (2) new and revised homework problems, and (3) an appendix introducing Simulink.

1. New Content. I have included new and enhanced sections on a number of topics. The primary additions include material on some MATLAB functions not included in previous editions (e.g., `fsolve, integrate, bvp4c`), some new applications of Monte Carlo for problems such as integration and optimization, and MATLAB's new way to pass parameters to function functions.
2. New Homework Problems. Most of the end-of-chapter problems have been modified, and a variety of new problems have been added. In particular, an effort has been made to include several new problems for each chapter that are more challenging and difficult than the problems in the previous edition.
3. I have developed a short primer on Simulink which I have my students read prior to covering that topic. Although I recognize that some professors may not choose to cover Simulink, I included it as a teaching aid for those that do.

Aside from the new material and problems, the fourth edition is very similar to the third. In particular, I have endeavored to maintain most of the features contributing to its pedagogical effectiveness including extensive use of worked examples and engineering and scientific applications. As with the previous edition, I have made a concerted effort to make this book as "student-friendly" as possible. Thus, I've tried to keep my explanations straightforward and practical.

Although my primary intent is to empower students by providing them with a sound introduction to numerical problem solving, I have the ancillary objective of making this introduction exciting and pleasurable. I believe that motivated students who enjoy engineering and science, problem solving, mathematics—and yes—programming, will ultimately make better professionals. If my book fosters enthusiasm and appreciation for these subjects, I will consider the effort a success.

Acknowledgments. Several members of the McGraw-Hill team have contributed to this project. Special thanks are due to Jolynn Kilburg, Thomas Scaife, Ph.D., Chelsea Haupt, Ph.D., and Jeni McAtee for their encouragement, support, and direction.

During the course of this project, the folks at The MathWorks, Inc., have truly demonstrated their overall excellence as well as their strong commitment to engineering and science education. In particular, Naomi Fernandes of The MathWorks, Inc., Book Program has been especially helpful and Jared Wasserman of the MathWorks Technical Support Department was of great help with technical questions.

The generosity of the Berger family has provided me with the opportunity to work on creative projects such as this book dealing with computing and engineering. In addition, my colleagues in the School of Engineering at Tufts, notably Masoud Sanayei, Babak Moaveni, Luis Dorfmann, Rob White, Linda Abriola, and Laurie Baise, have been very supportive and helpful.

Significant suggestions were also given by a number of colleagues. In particular, Dave Clough (University of Colorado–Boulder), and Mike Gustafson (Duke University) provided valuable ideas and suggestions. In addition, a number of reviewers provided useful feedback and advice including Karen Dow Ambtman (University of Alberta), Jalal Behzadi (Shahid Chamran University), Eric Cochran (Iowa State University), Frederic Gibou (University of California at Santa Barbara), Jane Grande-Allen (Rice University), Raphael Haftka (University of Florida), Scott Hendricks (Virginia Tech University), Ming Huang (University of San Diego), Oleg Igoshin (Rice University), David Jack (Baylor University), Se Won Lee (Sungkyunkwan University), Clare McCabe (Vanderbilt University), Eckart Meiburg (University of California at Santa Barbara), Luis Ricardez (University of Waterloo), James Rottman (University of California, San Diego), Bingjing Su (University of Cincinnati), Chin-An Tan (Wayne State University), Joseph Tipton (The University of Evansville), Marion W. Vance (Arizona State University), Jonathan Vande Geest (University of Arizona), Leah J. Walker (Arkansas State University), Qiang Hu (University of Alabama, Huntsville), Yukinobu Tanimoto (Tufts University), Henning T. Søgaard (Aarhus University), and Jimmy Feng (University of British Columbia).

It should be stressed that although I received useful advice from the aforementioned individuals, I am responsible for any inaccuracies or mistakes you may find in this book. Please contact me via e-mail if you should detect any errors.

Finally, I want to thank my family, and in particular my wife, Cynthia, for the love, patience, and support they have provided through the time I've spent on this project.

Steven C. Chapra
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PEDAGOGICAL TOOLS

Theory Presented as It Informs Key Concepts. The text is intended for Numerical Methods users, not developers. Therefore, theory is not included for "theory's sake," for example no proofs. Theory is included as it informs key concepts such as the Taylor series, convergence, condition, etc. Hence, the student is shown how the theory connects with practical issues in problem solving.

Introductory MATLAB Material. The text includes two introductory chapters on how to use MATLAB. Chapter 2 shows students how to perform computations and create graphs in MATLAB's standard command mode. Chapter 3 provides a primer on developing numerical programs via MATLAB M-file functions. Thus, the text provides students with the means to develop their own numerical algorithms as well as to tap into MATLAB's powerful built-in routines.

Algorithms Presented Using MATLAB M-files. Instead of using pseudocode, this book presents algorithms as well-structured MATLAB M-files. Aside from being useful computer programs, these provide students with models for their own M-files that they will develop as homework exercises.

Worked Examples and Case Studies. Extensive worked examples are laid out in detail so that students can clearly follow the steps in each numerical computation. The case studies consist of engineering and science applications which are more complex and richer than the worked examples. They are placed at the ends of selected chapters with the intention of (1) illustrating the nuances of the methods and (2) showing more realistically how the methods along with MATLAB are applied for problem solving.

Problem Sets. The text includes a wide variety of problems. Many are drawn from engineering and scientific disciplines. Others are used to illustrate numerical techniques and theoretical concepts. Problems include those that can be solved with a pocket calculator as well as others that require computer solution with MATLAB.

Useful Appendices and Indexes. Appendix A contains MATLAB commands, Appendix B contains M-file functions, and new Appendix C contains a brief Simulink primer.

Instructor Resources. Solutions Manual, Lecture PowerPoints, Text images in PowerPoint, M-files and additional MATLAB resources are available through Connect®.

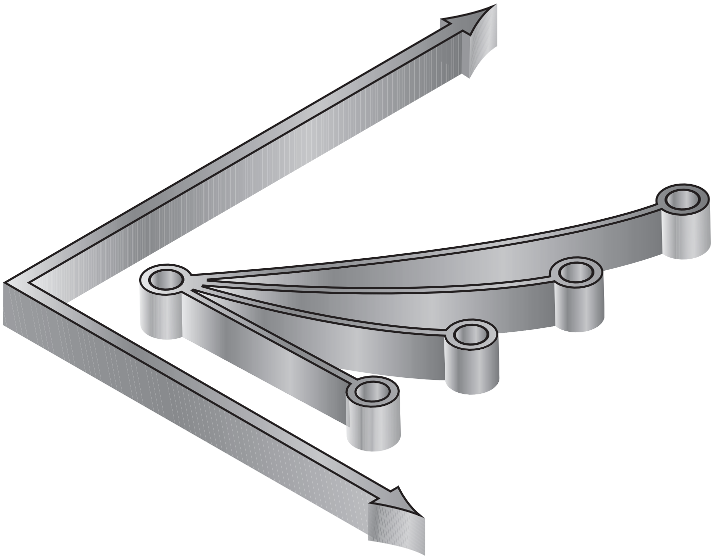
Modeling, Computers, and Error Analysis

1.1 MOTIVATION

What are numerical methods and why should you study them?

Numerical methods are techniques by which mathematical problems are formulated so that they can be solved with arithmetic and logical operations. Because digital computers excel at performing such operations, numerical methods are sometimes referred to as *computer mathematics*.

In the pre–computer era, the time and drudgery of implementing such calculations seriously limited their practical use. However, with the advent of fast, inexpensive digital computers, the role of numerical methods in engineering and scientific problem solving has exploded. Because they figure so prominently in much of our work, I believe that numerical methods should be a part of every engineer's and scientist's basic education. Just as we all must have solid foundations in the other areas of mathematics and science, we should also have a fundamental understanding of numerical methods. In particular, we should have a solid appreciation of both their capabilities and their limitations.



Beyond contributing to your overall education, there are several additional reasons why you should study numerical methods:

1. Numerical methods greatly expand the types of problems you can address. They are capable of handling large systems of equations, nonlinearities, and complicated geometries that are not uncommon in engineering and science and that are often impossible to solve analytically with standard calculus. As such, they greatly enhance your problem-solving skills.
2. Numerical methods allow you to use "canned" software with insight. During

your career, you will invariably have occasion to use commercially available prepackaged computer programs that involve numerical methods. The intelligent use of these programs is greatly enhanced by an understanding of the basic theory underlying the methods. In the absence of such understanding, you will be left to treat such packages as "black boxes" with little critical insight into their inner workings or the validity of the results they produce.

3. Many problems cannot be approached using canned programs. If you are conversant with numerical methods, and are adept at computer programming, you can design your own programs to solve problems without having to buy or commission expensive software.
4. Numerical methods are an efficient vehicle for learning to use computers. Because numerical methods are expressly designed for computer implementation, they are ideal for illustrating the computer's powers and limitations. When you successfully implement numerical methods on a computer, and then apply them to solve otherwise intractable problems, you will be provided with a dramatic demonstration of how computers can serve your professional development. At the same time, you will also learn to acknowledge and control the errors of approximation that are part and parcel of large-scale numerical calculations.
5. Numerical methods provide a vehicle for you to reinforce your understanding of mathematics. Because one function of numerical methods is to reduce higher mathematics to basic arithmetic operations, they get at the "nuts and bolts" of some otherwise obscure topics. Enhanced understanding and insight can result from this alternative perspective.

With these reasons as motivation, we can now set out to understand how numerical methods and digital computers work in tandem to generate reliable solutions to mathematical problems. The remainder of this book is devoted to this task.

1.2 PART ORGANIZATION

This book is divided into six parts. The latter five parts focus on the major areas of numerical methods. Although it might be tempting to jump right into this material, *Part One* consists of four chapters dealing with essential background material.

Chapter 1 provides a concrete example of how a numerical method can be employed to solve a real problem. To do this, we develop a *mathematical model* of a free-falling bungee jumper. The model, which is based on Newton's second law, results in an ordinary differential equation. After first using calculus to develop a closed-form solution, we then show how a comparable solution can be generated with a simple numerical method. We end the chapter with an overview of the major areas of numerical methods that we cover in Parts Two through Six.

Chapters 2 and 3 provide an introduction to the MATLAB® software environment. *Chapter 2* deals with the standard way of operating MATLAB by entering commands one at a time in the so-called *calculator,* or *command, mode*. This interactive mode provides a straightforward means to orient you to the environment and illustrates how it is used for common operations such as performing calculations and creating plots.

Chapter 3 shows how MATLAB's *programming mode* provides a vehicle for assembling individual commands into algorithms. Thus, our intent is to illustrate how MATLAB serves as a convenient programming environment to develop your own software.

Chapter 4 deals with the important topic of error analysis, which must be understood for the effective use of numerical methods. The first part of the chapter focuses on the *roundoff errors* that result because digital computers cannot represent some quantities exactly. The latter part addresses *truncation errors* that arise from using an approximation in place of an exact mathematical procedure.

1

Mathematical Modeling, Numerical Methods, and Problem Solving

CHAPTER OBJECTIVES

The primary objective of this chapter is to provide you with a concrete idea of what numerical methods are and how they relate to engineering and scientific problem solving. Specific objectives and topics covered are

- Learning how mathematical models can be formulated on the basis of scientific principles to simulate the behavior of a simple physical system.
- Understanding how numerical methods afford a means to generate solutions in a manner that can be implemented on a digital computer.
- Understanding the different types of conservation laws that lie beneath the models used in the various engineering disciplines and appreciating the difference between steady-state and dynamic solutions of these models.
- Learning about the different types of numerical methods we will cover in this book.

YOU'VE GOT A PROBLEM

Suppose that a bungee-jumping company hires you. You're given the task of predicting the velocity of a jumper (Fig. 1.1) as a function of time during the free-fall part of the jump. This information will be used as part of a larger analysis to determine the length and required strength of the bungee cord for jumpers of different mass.

You know from your studies of physics that the acceleration should be equal to the ratio of the force to the mass (Newton's second law). Based on this insight and your knowledge

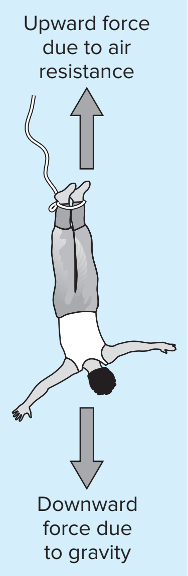

Upward force due to air resistance
Downward force due to gravity

FIGURE 1.1
Forces acting on a free-falling bungee jumper.

of physics and fluid mechanics, you develop the following mathematical model for the rate of change of velocity with respect to time,

$$\frac{dv}{dt} = g - \frac{c_d}{m}v^2$$

where v = downward vertical velocity (m/s), t = time (s), g = the acceleration due to gravity ($\cong$9.81 m/s^2), c_d = a lumped drag coefficient (kg/m), and m = the jumper's mass (kg). The drag coefficient is called "lumped" because its magnitude depends on factors such as the jumper's area and the fluid density (see Sec. 1.4).

Because this is a differential equation, you know that calculus might be used to obtain an analytical or exact solution for v as a function of t. However, in the following pages, we will illustrate an alternative solution approach. This will involve developing a computer-oriented numerical or approximate solution.

Aside from showing you how the computer can be used to solve this particular problem, our more general objective will be to illustrate (*a*) what numerical methods are and (*b*) how they figure in engineering and scientific problem solving. In so doing, we will also show how mathematical models figure prominently in the way engineers and scientists use numerical methods in their work.

1.1 A SIMPLE MATHEMATICAL MODEL

A *mathematical model* can be broadly defined as a formulation or equation that expresses the essential features of a physical system or process in mathematical terms. In a very general sense, it can be represented as a functional relationship of the form

$$\begin{matrix}\text{Dependent}\\ \text{variable}\end{matrix} = f\left(\begin{matrix}\text{independent}\\ \text{variables}\end{matrix}, \text{parameters}, \begin{matrix}\text{forcing}\\ \text{functions}\end{matrix}\right) \tag{1.1}$$

where the *dependent variable* is a characteristic that typically reflects the behavior or state of the system; the *independent variables* are usually dimensions, such as time and space, along which the system's behavior is being determined; the *parameters* are reflective of the system's properties or composition; and the *forcing functions* are external influences acting upon it.

The actual mathematical expression of Eq. (1.1) can range from a simple algebraic relationship to large complicated sets of differential equations. For example, on the basis of his observations, Newton formulated his second law of motion, which states that the time rate of change of momentum of a body is equal to the resultant force acting on it. The mathematical expression, or model, of the second law is the well-known equation

$$F = ma \tag{1.2}$$

where F is the net force acting on the body (N, or kg m/s^2), m is the mass of the object (kg), and a is its acceleration (m/s^2).

The second law can be recast in the format of Eq. (1.1) by merely dividing both sides by m to give

$$a = \frac{F}{m} \tag{1.3}$$

where a is the dependent variable reflecting the system's behavior, F is the forcing function, and m is a parameter. Note that for this simple case there is no independent variable because we are not yet predicting how acceleration varies in time or space.

Equation (1.3) has a number of characteristics that are typical of mathematical models of the physical world.

- It describes a natural process or system in mathematical terms.
- It represents an idealization and simplification of reality. That is, the model ignores negligible details of the natural process and focuses on its essential manifestations. Thus, the second law does not include the effects of relativity that are of minimal importance when applied to objects and forces that interact on or about the earth's surface at velocities and on scales visible to humans.
- Finally, it yields reproducible results and, consequently, can be used for predictive purposes. For example, if the force on an object and its mass are known, Eq. (1.3) can be used to compute acceleration.

Because of its simple algebraic form, the solution of Eq. (1.2) was obtained easily. However, other mathematical models of physical phenomena may be much more complex, and either cannot be solved exactly or require more sophisticated mathematical techniques than simple algebra for their solution. To illustrate a more complex model of this kind, Newton's second law can be used to determine the terminal velocity of a free-falling body near the earth's surface. Our falling body will be a bungee jumper (Fig. 1.1). For this case, a model can be derived by expressing the acceleration as the time rate of change of the velocity ($d\upsilon/dt$) and substituting it into Eq. (1.3) to yield

$$\frac{d\upsilon}{dt} = \frac{F}{m} \tag{1.4}$$

where υ is velocity (in meters per second). Thus, the rate of change of the velocity is equal to the net force acting on the body normalized to its mass. If the net force is positive, the object will accelerate. If it is negative, the object will decelerate. If the net force is zero, the object's velocity will remain at a constant level.

Next, we will express the net force in terms of measurable variables and parameters. For a body falling within the vicinity of the earth, the net force is composed of two opposing forces: the downward pull of gravity F_D and the upward force of air resistance F_U (Fig. 1.1):

$$F = F_D + F_U \tag{1.5}$$

If force in the downward direction is assigned a positive sign, the second law can be used to formulate the force due to gravity as

$$F_D = mg \tag{1.6}$$

where g is the acceleration due to gravity (9.81 m/s^2).

Air resistance can be formulated in a variety of ways. Knowledge from the science of fluid mechanics suggests that a good first approximation would be to assume that it is proportional to the square of the velocity,

$$F_U = -c_d v^2 \tag{1.7}$$

where c_d is a proportionality constant called the *lumped drag coefficient* (kg/m). Thus, the greater the fall velocity, the greater the upward force due to air resistance. The parameter c_d accounts for properties of the falling object, such as shape or surface roughness, that affect air resistance. For the present case, c_d might be a function of the type of clothing or the orientation used by the jumper during free fall.

The net force is the difference between the downward and upward force. Therefore, Eqs. (1.4) through (1.7) can be combined to yield

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2 \tag{1.8}$$

Equation (1.8) is a model that relates the acceleration of a falling object to the forces acting on it. It is a *differential equation* because it is written in terms of the differential rate of change (dv/dt) of the variable that we are interested in predicting. However, in contrast to the solution of Newton's second law in Eq. (1.3), the exact solution of Eq. (1.8) for the velocity of the jumper cannot be obtained using simple algebraic manipulation. Rather, more advanced techniques such as those of calculus must be applied to obtain an exact or analytical solution. For example, if the jumper is initially at rest ($v = 0$ at $t = 0$), calculus can be used to solve Eq. (1.8) for

$$v(t) = \sqrt{\frac{gm}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}}\, t\right) \tag{1.9}$$

where tanh is the hyperbolic tangent that can be either computed directly[1] or via the more elementary exponential function as in

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{1.10}$$

Note that Eq. (1.9) is cast in the general form of Eq. (1.1) where $v(t)$ is the dependent variable, t is the independent variable, c_d and m are parameters, and g is the forcing function.

EXAMPLE 1.1 Analytical Solution to the Bungee Jumper Problem

Problem Statement. A bungee jumper with a mass of 68.1 kg leaps from a stationary hot air balloon. Use Eq. (1.9) to compute velocity for the first 12 s of free fall. Also determine the terminal velocity that will be attained for an infinitely long cord (or alternatively, the jumpmaster is having a particularly bad day!). Use a drag coefficient of 0.25 kg/m.

[1] MATLAB allows direct calculation of the hyperbolic tangent via the built-in function `tanh(x)`.

Solution. Inserting the parameters into Eq. (1.9) yields

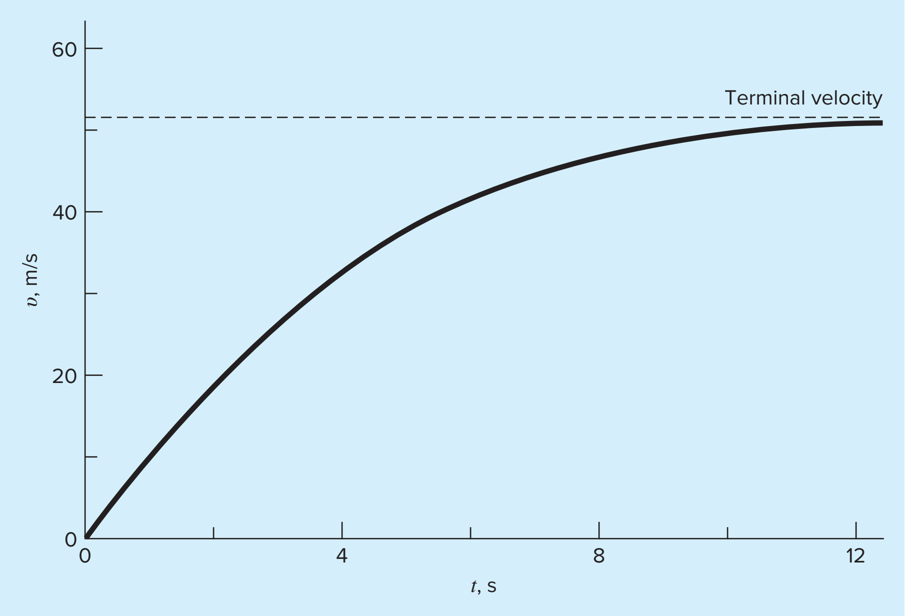
$$v(t) = \sqrt{\frac{9.81(68.1)}{0.25}}\tanh\left(\sqrt{\frac{9.81(0.25)}{68.1}}\,t\right) = 51.6938\tanh(0.18977t)$$

which can be used to compute

t, s	*v*, m/s
0	0
2	18.7292
4	33.1118
6	42.0762
8	46.9575
10	49.4214
12	50.6175
∞	51.6938

According to the model, the jumper accelerates rapidly (Fig. 1.2). A velocity of 49.4214 m/s (about 110 mi/hr) is attained after 10 s. Note also that after a sufficiently

FIGURE 1.2
The analytical solution for the bungee jumper problem as computed in Example 1.1. Velocity increases with time and asymptotically approaches a terminal velocity.



long time, a constant velocity, called the *terminal velocity,* of 51.6983 m/s (115.6 mi/hr) is reached. This velocity is constant because, eventually, the force of gravity will be in balance with the air resistance. Thus, the net force is zero and acceleration has ceased.

Equation (1.9) is called an *analytical* or *closed-form solution* because it exactly satisfies the original differential equation. Unfortunately, there are many mathematical models that cannot be solved exactly. In many of these cases, the only alternative is to develop a numerical solution that approximates the exact solution.

Numerical methods are those in which the mathematical problem is reformulated so it can be solved by arithmetic operations. This can be illustrated for Eq. (1.8) by realizing that the time rate of change of velocity can be approximated by (Fig. 1.3):

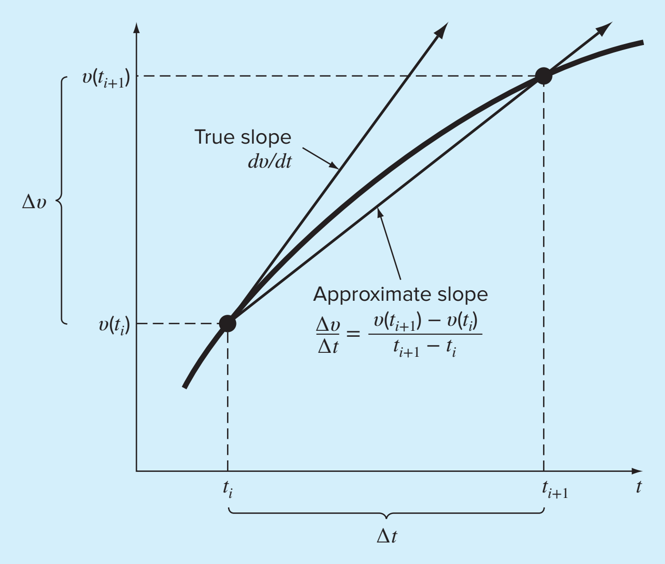
$$\frac{dv}{dt} \cong \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} \tag{1.11}$$

where Δv and Δt are differences in velocity and time computed over finite intervals, $v(t_i)$ is velocity at an initial time t_i, and $v(t_{i+1})$ is velocity at some later time t_{i+1}. Note that $dv/dt \cong \Delta v/\Delta t$ is approximate because Δt is finite. Remember from calculus that

$$\frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

Equation (1.11) represents the reverse process.

FIGURE 1.3
The use of a finite difference to approximate the first derivative of v with respect to t.



Equation (1.11) is called a *finite-difference approximation* of the derivative at time t_i. It can be substituted into Eq. (1.8) to give

$$\frac{\upsilon(t_{i+1}) - \upsilon(t_i)}{t_{i+1} - t_i} = g - \frac{c_d}{m}\upsilon(t_i)^2$$

This equation can then be rearranged to yield

$$\upsilon(t_{i+1}) = \upsilon(t_i) + \left[g - \frac{c_d}{m}\upsilon(t_i)^2\right](t_{i+1} - t_i) \tag{1.12}$$

Notice that the term in brackets is the right-hand side of the differential equation itself [Eq. (1.8)]. That is, it provides a means to compute the rate of change or slope of υ. Thus, the equation can be rewritten more concisely as

$$\upsilon_{i+1} = \upsilon_i + \frac{d\upsilon_i}{dt}\Delta t \tag{1.13}$$

where the nomenclature υ_i designates velocity at time t_i, and $\Delta t = t_{i+1} - t_i$.

We can now see that the differential equation has been transformed into an equation that can be used to determine the velocity algebraically at t_{i+1} using the slope and previous values of υ and t. If you are given an initial value for velocity at some time t_i, you can easily compute velocity at a later time t_{i+1}. This new value of velocity at t_{i+1} can in turn be employed to extend the computation to velocity at t_{i+2} and so on. Thus at any time along the way,

New value = old value + slope × step size

This approach is formally called *Euler's method*. We'll discuss it in more detail when we turn to differential equations later in this book.

EXAMPLE 1.2 Numerical Solution to the Bungee Jumper Problem

Problem Statement. Perform the same computation as in Example 1.1 but use Eq. (1.12) to compute velocity with Euler's method. Employ a step size of 2 s for the calculation.

Solution. At the start of the computation ($t_0 = 0$), the velocity of the jumper is zero. Using this information and the parameter values from Example 1.1, Eq. (1.12) can be used to compute velocity at $t_1 = 2$ s:

$$\upsilon = 0 + \left[9.81 - \frac{0.25}{68.1}(0)^2\right] \times 2 = 19.62 \text{ m/s}$$

For the next interval (from $t = 2$ to 4 s), the computation is repeated, with the result

$$\upsilon = 19.62 + \left[9.81 - \frac{0.25}{68.1}(19.62)^2\right] \times 2 = 36.4137 \text{ m/s}$$

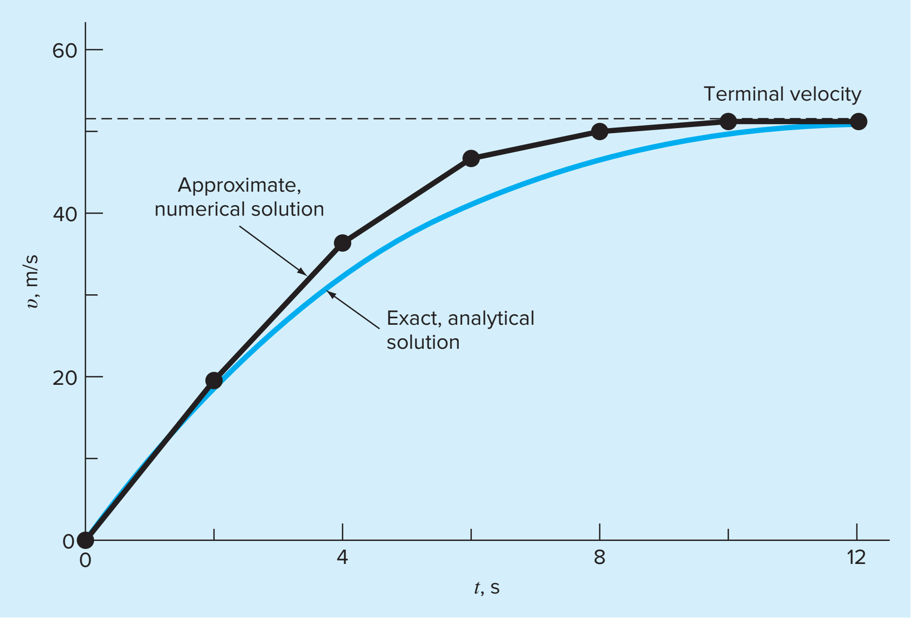


FIGURE 1.4
Comparison of the numerical and analytical solutions for the bungee jumper problem.

The calculation is continued in a similar fashion to obtain additional values:

t, s	v, m/s
0	0
2	19.6200
4	36.4137
6	46.2983
8	50.1802
10	51.3123
12	51.6008
∞	51.6938

The results are plotted in Fig. 1.4 along with the exact solution. We can see that the numerical method captures the essential features of the exact solution. However, because we have employed straight-line segments to approximate a continuously curving function, there is some discrepancy between the two results. One way to minimize such discrepancies is to use a smaller step size. For example, applying Eq. (1.12) at 1-s intervals results in a smaller error, as the straight-line segments track closer to the true solution. Using hand calculations, the effort associated with using smaller and smaller step sizes would make such numerical solutions impractical. However, with the aid of the computer, large numbers of calculations can be performed easily. Thus, you can accurately model the velocity of the jumper without having to solve the differential equation exactly.

As in Example 1.2, a computational price must be paid for a more accurate numerical result. Each halving of the step size to attain more accuracy leads to a doubling of the number of computations. Thus, we see that there is a trade-off between accuracy and computational effort. Such trade-offs figure prominently in numerical methods and constitute an important theme of this book.

1.2 CONSERVATION LAWS IN ENGINEERING AND SCIENCE

Aside from Newton's second law, there are other major organizing principles in science and engineering. Among the most important of these are the *conservation laws*. Although they form the basis for a variety of complicated and powerful mathematical models, the great conservation laws of science and engineering are conceptually easy to understand. They all boil down to

$$\text{Change} = \text{increases} - \text{decreases} \tag{1.14}$$

This is precisely the format that we employed when using Newton's law to develop a force balance for the bungee jumper [Eq. (1.8)].

Although simple, Eq. (1.14) embodies one of the most fundamental ways in which conservation laws are used in engineering and science—that is, to predict changes with respect to time. We will give it a special name—the *time-variable* (or *transient*) computation.

Aside from predicting changes, another way in which conservation laws are applied is for cases where change is nonexistent. If change is zero, Eq. (1.14) becomes

$$\text{Change} = 0 = \text{increases} - \text{decreases}$$

or

$$\text{Increases} = \text{decreases} \tag{1.15}$$

Thus, if no change occurs, the increases and decreases must be in balance. This case, which is also given a special name—the *steady-state* calculation—has many applications in engineering and science. For example, for steady-state incompressible fluid flow in pipes, the flow into a junction must be balanced by flow going out, as in

$$\text{Flow in} = \text{flow out}$$

For the junction in Fig. 1.5, the balance that can be used to compute that the flow out of the fourth pipe must be 60.

For the bungee jumper, the steady-state condition would correspond to the case where the net force was zero or [Eq. (1.8) with $dv/dt = 0$]

$$mg = c_d v^2 \tag{1.16}$$

Thus, at steady state, the downward and upward forces are in balance and Eq. (1.16) can be solved for the terminal velocity

$$v = \sqrt{\frac{gm}{c_d}}$$

Although Eqs. (1.14) and (1.15) might appear trivially simple, they embody the two fundamental ways that conservation laws are employed in engineering and science. As such, they will form an important part of our efforts in subsequent chapters to illustrate the connection between numerical methods and engineering and science.

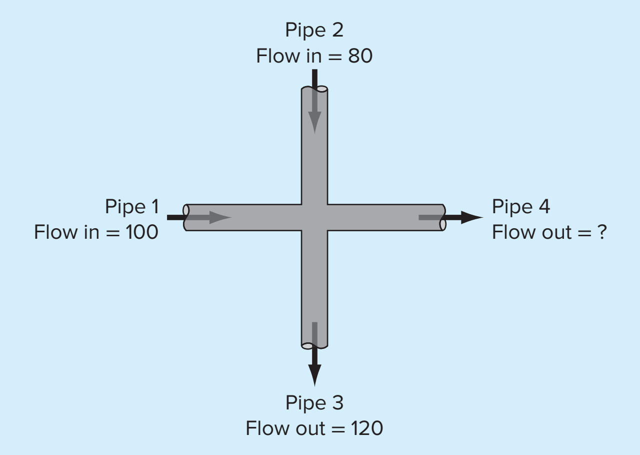


FIGURE 1.5
A flow balance for steady incompressible fluid flow at the junction of pipes.

Table 1.1 summarizes some models and associated conservation laws that figure prominently in engineering. Many chemical engineering problems involve mass balances for reactors. The mass balance is derived from the conservation of mass. It specifies that the change of mass of a chemical in the reactor depends on the amount of mass flowing in minus the mass flowing out.

Civil and mechanical engineers often focus on models developed from the conservation of momentum. For civil engineering, force balances are utilized to analyze structures such as the simple truss in Table 1.1. The same principles are employed for the mechanical engineering case studies to analyze the transient up-and-down motion or vibrations of an automobile.

Finally, electrical engineering studies employ both current and energy balances to model electric circuits. The current balance, which results from the conservation of charge, is similar in spirit to the flow balance depicted in Fig. 1.5. Just as flow must balance at the junction of pipes, electric current must balance at the junction of electric wires. The energy balance specifies that the changes of voltage around any loop of the circuit must add up to zero.

It should be noted that there are many other branches of engineering beyond chemical, civil, electrical, and mechanical. Many of these are related to the Big Four. For example, chemical engineering skills are used extensively in areas such as environmental, petroleum, and biomedical engineering. Similarly, aerospace engineering has much in common with mechanical engineering. I will endeavor to include examples from these areas in the coming pages.

1.3 NUMERICAL METHODS COVERED IN THIS BOOK

Euler's method was chosen for this introductory chapter because it is typical of many other classes of numerical methods. In essence, most consist of recasting mathematical operations into the simple kind of algebraic and logical operations compatible with digital computers. Figure 1.6 summarizes the major areas covered in this text.